ECE 443 Lab 1

September 2, 2011

1 Discrete Fourier Transform

The discrete Fourier transform (DFT) is an operation that takes sequences of length N and returns sequences of length N ($\mathcal{F} : \mathbb{C}^N \to \mathbb{C}^N$).

$$\mathcal{F}\{x\} = X[k] = \sum_{n=0}^{N-1} x[n] w_N^{-kn}, \ k = 0, 1, \ldots, N-1,$$

where $w_N = e^{2\pi j/N}$. Sometimes the powers of w_N are called "twiddle factors". The inverse transform (IDFT) is

$$\mathcal{F}^{-1}\{X\} = x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] w_N^{kn}, \ n = 0, 1, \ldots, N-1,$$

Notice the similarity of the forward DFT and the inverse DFT, and their similarity to the continuous time Fourier transform (CTFT) and the Fourier series equations. Like Fourier series, the time domain signal is assumed to be periodic, but in the case of the DFT, the frequency domain signal is also periodic.

DFTs are important in practice because there exist efficient algorithms (called fast Fourier transforms, or FFTs) for computing them. In fact, most mathematical software packages will simply call the DFT operation an FFT (e.g. in Matlab one computes DFTs and their inverses with the `fft()` and `ifft()` commands). We will be using Matlab, which has the unfortunate convention that array indexes start at 1.[1] This often makes for a lot of $(n-1)$ or $(n+1)$ expressions in your Matlab code. For reference, $x[n] = \texttt{x(n}-\texttt{1)}$ and $X[k] = \texttt{X(k}-\texttt{1)}$ when comparing the above formulas to Matlab indexing.

In general, a real input to the DFT will give a complex result, but the symmetry of being real ($x = x^*$) is transformed to a similar symmetry

$$X[k] = X^*[N-k].$$

In other words, the frequency domain signal is equal to a conjugated, reversed version of itself. In plots, this means that there will be a vertical mirrorlike symmetry. Only half of the sequence is important (because we can generate one from the other).

Consider $x[n] = x(t_n) = x(nT_s)$ to be samples of a continuous time signal $x(t)$. We can approximate the CTFT with a DFT:

$$T_s X[k] = \sum_{n=0}^{N-1} x(t_n) e^{-2\pi j f_k t_n} T_s \approx \int_0^{NT_s} x(t) e^{-2\pi j f_k t} dt \approx X(f_k),$$

where $f_k = k/(NT_s)$. We must choose the sampling frequency $f_s = 1/T_s$ to be high enough to minimize aliasing, and choose N large enough that we capture the important parts of $x(t)$ in time (many signals

[1] Why is this subjective choice objectively wrong? See `http://www.cs.utexas.edu/users/EWD/transcriptions/EWD08xx/EWD831.html`.

of interest approach zero as $|t| \to \infty$). We can shift $x(t)$ in time if needed, which will effectively shift the integration limits. For example if $g(t) = e^{-|t|}$, we would choose $x(t) = g(t - NT_s/2)$, which gives $X(f) = G(f)e^{-2\pi j f N T_s/2}$.

2 DFT Examples

Consider the following Matlab code:

```
T = 1;
N = 64;
f = 3/N;
t = [0:T:N-1];
x = sin(2*pi*f*t);
X = fft(x);
stem(t,abs(X));
```

Try it and explain what the plot is showing.

Set $f = 3.5/N$. Why does it look different?

Set $f = 40/N$. Explain the difference.

Try setting x to be a sum of sinusoids with different phases, amplitudes, and frequencies (both integer and non-integer multiples of $1/N$), and comment on the spectrum plots.

Write some code that uses the DFT to approximate the CTFT of $e^{-2t}u(t)$. For reference, $X(f) = \frac{1}{2+2\pi j f}$. Compare the approximation with the true CTFT. Plot $|X(f_k) - TX[k]|$ vs. f_k.

3 Spectrum Analyzer

Consider $x(t) = e^{2\pi j f t}$ as a time signal with single fixed frequency f. We will see how the DFT performs as a spectrum analyzer. Let $x[n] = x(nT) = e^{2\pi j f n T}$. It can be shown that

$$X[k] = e^{2\pi j (f - f_k) T (N-1)/2} \frac{\sin(\pi N (f - f_k) T)}{\sin(\pi (f - f_k) T)}.$$

Consider

$$g_N(\lambda) = \frac{\sin(N\lambda)}{\sin(\lambda)}.$$

Plot $g_N(\lambda)$ for $0 \leq \lambda \leq 4\pi$ with $N = 32$. Clearly $|X[k]| = |g_N(\pi(f - f_k)T)|$. Also note that $g_N(2\pi f_m) = N$ when m is an integer multiple of N, and $g(2\pi f_m) = 0$ otherwise, and that DFTs are linear operations. Can you use this information to explain some earlier results in the lab?